

Comment on “Thermal Hall Effect and Geometry with Torsion”

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Abstract

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This is a comment on a paper by Gromov and Abanov [1]. We will show that there is an inconsistency which renders the results untenable. In their paper [1] a complex scalar field theory which is invariant under space and time translations is coupled with background curvature in the torsional Newton - Cartan (NC) formulation. They introduced the vielbeins E_a^μ where $a = 0, A; A = 1, 2$ and $\mu = 0, i; i = 1, 2$, where a and μ respectively label the tangent space and the curved space coordinates. The derivatives with respect to these coordinates are related by appropriate vielbeins E_a^μ

$$\partial_a \rightarrow E_a^\mu \partial_\mu, \quad (1)$$

The basis one-forms e_μ^a are then defined so that

$$e_\mu^a E_b^\mu = \delta_b^a, \quad e_\mu^a E_a^\nu = \delta_\nu^\mu \quad (2)$$

The NC geometry is characterized by the authors by constructing a degenerate metric

$$h^{\mu\nu} = \delta^{AB} E_A^\mu E_B^\nu \quad (3)$$

and an 1-form by

$$n_\mu = e_\mu^0. \quad (4)$$

They are defined to satisfy the NC geometry requirements

$$E_0^\mu n_\mu = 1, \quad h^{\mu\nu} n_\nu = 0. \quad (5)$$

After defining the NC geometry in the general way by equations (3), (2) and (5) from the vielbeins the authors go on to assume that in a particular parametrization the metric takes the following form:

$$h^{\mu\nu} = \begin{pmatrix} \frac{n^2}{n_0^2} & -\frac{n^i}{n_0} \\ -\frac{n^i}{n_0} & h^{ij} \end{pmatrix}, \quad (6)$$

where $n^i = h^{ij}n_j$ and $n^2 = n_in_jh^{ij}$. It may be observed that the authors are considering NC spacetime to be foliated in spacelike hypersurfaces where h^{ij} is a nondegenerate metric attributed to the hypersurface. In other words, the authors are using the Galilean frame. This is supported by their declaration – “Notice, that the spatial part of the metric h^{ij} is a (inverse) metric on a fixed time slice, it is symmetric and invertible.”

We first observe that for the NC spacetime subject to metric compatibility,

$$n_\mu = \partial_\mu t, \quad (7)$$

for some time function t . This has nothing to do with torsion or no torsion. To see this note that due to metric compatibility, $\nabla_\rho \tau_\mu = 0$, where

$$\nabla_\rho \tau_\mu = \partial_\rho \tau_\mu + \Gamma_{\rho\mu}^\lambda \tau_\lambda \quad (8)$$

is the covariant derivative and $\Gamma_{\rho\mu}^\lambda$ is the connection. On account of the metric compatibility,

$$\nabla_\rho \tau_\mu - \nabla_\mu \tau_\rho = 0 \quad (9)$$

Then, on account of the Poincare lemma there exists (at least locally) a function t of the spacetime coordinates such that,

$$n_\mu = \nabla_\mu t = \partial_\mu t \quad (10)$$

In case of the torsional NC geometry. (7) is only locally true. But that is sufficient for us.

The NC spacetime can be foliated in spacelike hypersurfaces in a unique way, using t as the affine parameter [2, 3]. In case of standard (torsionless) NC geometry t is a global function. Otherwise it is local. The “fixed time slice” considered by the authors is everywhere orthogonal to n_ν . On it is defined the nondegenerate spatial metric h^{ij} . The contravariant and covariant components of 3 vectors can be related as

$${}^{(3)}V^i = h^{ij}V_j \quad (11)$$

In other words we have to choose

$$x^0 = t. \quad (12)$$

as our coordinate time. This means working in the adapted coordinates [5] [2, 3]. From (4) and (7) we obtain

$$e_\mu^0 = \partial_\mu t = e_\mu^b \partial_b t, \quad (13)$$

which leads to (12). Recalling that the spacelike hypersurface is orthogonal to the direction of local time flow, we obtain, from the above relations, the following parametrisation,

$$e_\mu^0 = (1, 0, 0, 0). \quad (14)$$

Our point is now to show that the above results are inconsistent with the choice of metric (6). A simple calculation is sufficient to prove it. From the above definitions we find

$$\begin{aligned} n^2 &= n_in_jh^{ij} = n_in_jE_A^iE_A^j = (n_iE_A^i)(n_jE_A^j) \\ &= (n_\mu E_A^\mu - n_0 E_A^0)(n_\nu E_A^\nu - n_0 E_A^0) \\ &= n_\mu n_\nu h^{\mu\nu} - 2n_0 E_A^0 n_\mu E_A^\mu + (n_0 E_A^0)^2 \\ &= -2n_0 E_A^0 n_\mu E_A^\mu + (n_0 E_A^0)^2 \end{aligned} \quad (15)$$

where we have used (3) and (5).

Now, on the other hand, (6) and (3) give

$$n^2 = n_0^2 h^{00} = (n_0 E_A^0)^2. \quad (16)$$

Combining (15) and (16) we are left with two possibilities:

1. $n_\mu E_A^\mu = 0$, or
2. $n_0 E_A^0 = 0$

The first condition is superfluous since it is identical to the second one. To see this we use (4) and (14) to find,

$$n_\mu E_A^\mu = e_\mu^0 E_A^\mu = E_A^0 = 0. \quad (17)$$

On the other hand the second condition also leads to the same result, since n_0 is non-vanishing.

As a consequence $h^{00} = 0$ and $n^2 = 0$. This means in turn $n^i = 0$. Thus $h^{0i} = h^{i0} = 0$. These conclusions follow in order to satisfy the second condition in (5). We are thus led to the following structure of the metric

$$h^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & h^{ij} \end{pmatrix}. \quad (18)$$

This metric has appeared in earlier studies [4, 5] on NC geometry.

In fact, that the construction of the metric in [1] leads to (18) can be understood from much more simple considerations. Look at the vector n^i on the 'fixed time slice'. We construct $\xi^\mu = (0, n^i)$. It is evident that ξ^μ is a space like vector. Then it is also undeniable that $\xi^\mu = h^{\mu\nu} \lambda_\nu$, where, $\lambda_\mu = (0, n_i)$. From the metric properties of the NC geometry, $h^{\mu\nu} \lambda_\mu \lambda_\nu = 0$ i.e $n^2 = h^{ij} n_i n_j = 0$. The $(0, 0)$ element of the authors proposed matrix vanishes. Again, since h^{ij} is invertible, $n^2 = 0$ implies $n_i = 0$.

To conclude, we show that the structure (6) is inconsistent with the particular parametrization chosen in [1]. Consistency is recovered by the choice 18. The key element is that the various structures assumed by the authors lead to the adapted coordinate $x^0 = t$, where t defines a Galilean frame [2, 3].

Since the subsequent analysis in [1] is done by using the metric (6), the results become untenable. Naturally, the physical conclusions drawn from the calculations based on (6) are questionable. .

References

- [1] A. Gromov and A. G. Abanov, Phys. Rev. Lett. 114, 016802 (2015).
- [2] H.P. Kunzle, Gen. Relativ. Gravit **7**, 445 (1976).
- [3] J. Christian, Phys. Rev **56**, 4844 (1997).
- [4] R. Banerjee, A. Mitra, P. Mukherjee, Class. Quantum Grav. 32, 045010 (2015).
- [5] Roel Andringa , Eric Bergshoeff , Sudhakar Panda and Mees de Roo, Class. Quantum Grav. 28 105011 (2011).